First Mid-Term

CS 599: Autonomous Cyber-Physical Systems

Instructor: Jyotirmoy V. Deshmukh

Assigned: March 22, 2019. Due in class: **April 1, 2019**

Instructions:

1. Please feel free to refer to any material that you deem fit.
2. Discussions among fellow students are generally not encouraged. If you do wish to ask questions, please do it on Slack or through email where Nicole or I will answer. Please adhere to the academic integrity policy.
3. We prefer that you turn in a pdf by email before class to both Nicole and me. You can also hand in a printed/handwritten copy in class.

Problem 1. [15 points] Consider the following dynamical system:

$$\begin{bmatrix} \dot{x_1} \\ \dot{x_2} \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 2x_2 - 4x_1^3 \end{bmatrix} \tag{1}$$

Let $\mathbf{x} = (x_1, x_2)$. Recall that the Lyapunov function of a system allows you to check if a system is asymptotically stable at a given equilibrium point. Observe that $\mathbf{0} = (0, 0)$ is an equilibrium point of the above system. A function $V(\mathbf{x})$ is a Lyapunov function of a system $\dot{\mathbf{x}} = f(\mathbf{x})$ if the following conditions hold over the domain of interest D:

1. $\forall \mathbf{x} \in D : (\mathbf{x} \neq \mathbf{0}) \implies V(\mathbf{x}) > 0$,
2. $\mathbf{x} = \mathbf{0} \implies V(\mathbf{x}) = 0$
3. $\forall \mathbf{x} \in D : (\mathbf{x} \neq \mathbf{0}) \implies \frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x}) < 0$
4. $\mathbf{x} = \mathbf{0} \implies \frac{\partial V}{\partial \mathbf{x}} f(\mathbf{x}) = 0$.

(a) [10 points] Which of the following functions are valid Lyapunov functions for the system, and why?

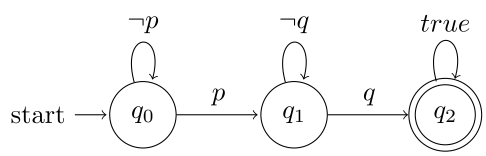
1. $V_1(\mathbf{x}) = x_1^2 + x_2^2$
2. $V_2(\mathbf{x}) = 4x_1^2 + 2x_2^2 + 4x_1^4$

(b) [5 points] Recall that the degree of a multi-variate monomial is the sum of the exponents of the variables appearing in that monomial. E.g., $d(x_1^3 x_2) = 4$. The degree of a polynomial is equal to the highest degree of any monomial that constitutes the polynomial. Can $V(\mathbf{x})$ have an odd degree and still be a Lyapunov function? If yes, give an example. If no, why not?

Problem 2. [15 points]

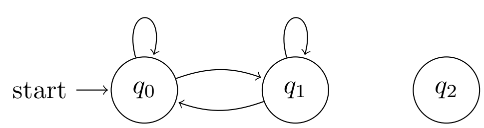
Recall the English language meanings of LTL operators **G** (always), **F** (eventually), **X** (next step) and **U** (until). Let lowercase letters represent propositions.

(a) [5 points] Recall that a Büchi automaton accepts a given infinite string only if a final state appears infinitely often on the sequence of states visited by the automaton when reading the string. Answer true or false: The following Büchi automaton corresponds to the LTL property $\mathbf{F}(p \wedge \mathbf{XF}q)$.



(b) [5 points] What would you change in the above automaton to make it accept strings corresponding to the following LTL property: $\mathbf{F}(p \wedge \mathbf{FG}q)$?

(c) [5 points] Add appropriate labels and mark the final state in the following Büchi automaton to ensure that the language accepted by the automaton is the same as that of the LTL property $\mathbf{G}(p \vee \mathbf{F}q)$:



Problem 3. [15 points] In this problem, we are going to perform the calculations that a Kalman filter performs manually. Consider the following discretized dynamics modeling the motion of a point mass at constant acceleration:

$$\begin{bmatrix} x_{k+1} \\ v_{k+1} \\ a_{k+1} \end{bmatrix} = \begin{bmatrix} 1 & \Delta & 0 \\ 0 & 1 & \Delta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_k \\ v_k \\ a_k \end{bmatrix} + \begin{bmatrix} w_{1k} \\ w_{2k} \\ w_{3k} \end{bmatrix} \tag{2}$$

Let the time increment $\Delta = 1$. Note that this system does not have any inputs. We use $\mathbf{x} = (x, v, a)$ as an abbreviation, and $\mathbf{w} = (w_1, w_2, w_3)$. At time k, an observation is made, subject to observation noise modeled by the variable v, and with $H = [1\ 0\ 0]$.

$$z_k = H \begin{bmatrix} x_k \\ v_k \\ a_k \end{bmatrix} + v_k$$

Recall that $a \sim \mathcal{N}(\mu, \sigma^2)$ means that a is a random variable normally distributed with mean μ and variance σ^2, and $\mathbf{a} \sim \mathcal{N}(\mu, \Sigma)$ means that a is a randomly distributed vector with mean μ and covariance Σ. Let $\mathbf{w} \sim \mathcal{N}(0, Q)$, where:

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Also, $v \sim \mathcal{N}(0, \sigma^2)$, where $\sigma^2 = 4$.

Starting at time 1, show what the Kalman filter estimates as the position using its two-step prediction and update equations. For your reference, the prediction and update equations for this case are as given by the following equations:

Prediction.

1. $\hat{\mathbf{x}}_{k|k-1} = A\hat{\mathbf{x}}_{k-1|k-1}$
2. $P_{k|k-1} = AP_{k-1|k-1}A^T + Q$

Update.

1. $y_k = z_k - H\hat{\mathbf{x}}_{k|k-1}$
2. $s_k = \sigma^2 + HP_{k|k-1}H^T$
3. $K_k = \frac{P_{k|k-1}H^T}{s_k}$
4. $\hat{\mathbf{x}}_{k|k} = \hat{\mathbf{x}}_{k|k-1} + K_k y_k$
5. $P_{k|k} = (I - K_kH)P_{k|k-1}(I - K_kH)^T + \sigma^2 K_k K_k^T$

(a) [10 points] Let $P_{0|0}$ be given as follows:

$$P_{0|0} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate the estimates of the position (i.e. $\hat{x}_{k|k}$) made by the Kalman filter for $k = 1$ to 5. Let $x_0 = 0$, and $v_0 = 10$. Let z be the actual measurements as specified by the table below. Please feel free to use calculators or Matlab.

k	$\hat{\mathbf{x}}_{k\|k}$	z_k
0	(0,10,1)	–
1		11
2		20
3		34
4		44
5		60

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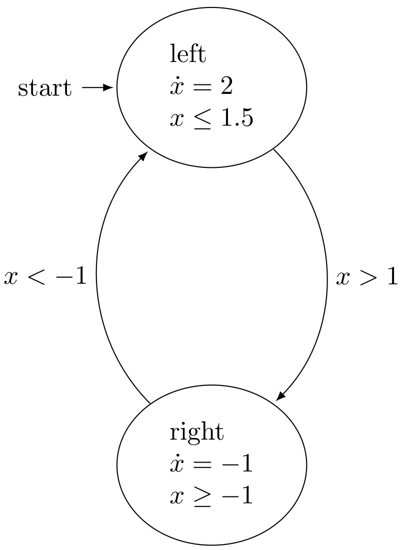
(b) [5 points] In the previous question, what would happen if there is a lot of uncertainty in the process model compared to the observation noise? I.e. the entries in the Q matrix are much larger compared to σ^2? On the other hand, What would happen if σ^2 is much larger compared to the entries in Q? You do not have to do any computation!

Problem 4. [15 points] Consider the hybrid process shown below:



(a) [5 points] Assuming you start in state (x = 0) and mode `left`, show an execution of this hybrid process that has at least two continuous transitions and two discrete transitions.

(b) [5 points] what is the maximum time the hybrid process can stay in any one mode (left/right)? Which mode?

(c) [5 points]

Suppose we replace the guard $x < -1$ by $x < -\epsilon$, and $x > 1$ by $x > \epsilon$, for some $\epsilon > 0$. Also, we replace the mode-invariants $x \leq 1.5$ and $x \geq -1$ by $x \leq 1.5\epsilon$ and $x \geq -\epsilon$ respectively. What happens as $\epsilon \to 0$?

Problem 5. [10 points] Consider an asynchronous process with two tasks:

$$\begin{array}{lccl} T_x : & (y > 0) & \to & x := (x+1) \bmod 3; \\ & & & y := 0 \\ \\ T_y : & (x < 2) & \to & y := (y+1) \bmod 3; \\ & & & x := (x+1) \mathrm{mod}\ 3; \end{array}$$

(a) [5 points] Starting with the initial state $x \mapsto 0, y \mapsto 0$, show the underlying transition system of this process.

(b) [5 points] Does the above process satisfy the CTL property $\mathbf{AG}(\neg((x = 0) \wedge (y = 2)))$?

(c) [Extra credit: 10 points] Does this transition system satisfy the LTL property $\mathbf{F}(x = 2)$ under strong fairness assumptions? (I.e. under the assumption that when the guards $(y > 0)$ and $(x < 2)$ are infinitely often enabled, the scheduler will respectively execute T_x and T_y infinitely often).

Problem 6. [15 points] Consider the discrete-time sampled signals given below:

t	x_1	x_2
0	1	0
0.5	0	0.3
1.0	1	-0.1
1.5	1	-3
2.5	0	1

Let $\mathbf{x} = (x_1, x_2)$. Let φ, ψ be STL formulas as given below:

- $\psi \equiv \mathbf{G}_{[0,2]} \left((x_1 \geq 0.5) \implies \mathbf{F}_{[0,0.5]}(x_2 \leq 0) \right)$
- $\varphi \equiv \mathbf{G}_{[0,0.5]} \mathbf{F}_{[0,0.5]}(x_1 \geq 0)$

Assume that the continuous time signal $\mathbf{x}$ can be obtained by constant interpolation between the sample points. In other words, given sampling times $t_0, t_1, \ldots, \forall i, \forall t \in [t_i, t_{i+1}) : \mathbf{x}(t) = \mathbf{x}(t_i)$. (i.e. "sample and hold" semantics).
(a) [5 points] Compute the robust satisfaction value $\rho(\psi, \mathbf{x}, 0)$. Does $\mathbf{x} \models \psi$?

(b) [5 points] Compute the robust satisfaction value $\rho(\varphi, \mathbf{x}, 0)$. Does $\mathbf{x} \models \varphi$?

(c) [5 points] Suppose that the continuous time signal $\mathbf{x}$ is obtained by linearly interpolating between sample points. I.e., $\forall t \in [t_i, t_{i+1}]$, $\mathbf{x}(t) = \mathbf{x}(t_i) + t\frac{\mathbf{x}(t_{i+1}) - \mathbf{x}(t_i)}{t_{i+1} - t_i}$. Will the robust satisfaction value $\rho(\varphi, \mathbf{x}, 0)$ remain the same? If yes, explain why. If no, what is the changed value?

Problem 7. [5 points] Consider the following transition probability matrix for a Markov chain:

	q_0	q_1	q_2	q_3
q_0	0.1	0.2	0.3	0.4
q_1	0.5	0.5	0	0
q_2	0	0.9	0.1	0
q_3	0.3	0.2	0.5	0

Assume that the Markov chain can start in any state with equal probability (0.25). What is the probability that if the Markov chain is in state q_0, then within at most 3 steps, it will be in state q_0 again? Feel free to use Matlab for performing the computations.

Problem 8. [5 points]
(a) What is the one thing that you really enjoyed learning in the course so far?

(b) What is the one thing that you did not enjoy learning about in this course so far?